Journal of Intelligent & Fuzzy Systems 26 (2014) 1799–1811
DOI:10.3233/IFS-130859
IOS Press

Uncertainty measure of Atanassov's intuitionistic fuzzy $\mathcal{T}$ equivalence information systems

Weihua Xu*, Yufeng Liu and Wenxin Sun
School of Mathematics and Statistics, Chongqing University of Technology, Chongqing, P.R. China

Abstract. Atanassov's intuitionistic fuzzy $\mathcal{T}$ equivalence information systems are natural extensions of fuzzy $\mathcal{T}$ equivalence information systems. The aim of this paper is to investigate the uncertainty measures of knowledge in Atanassov's intuitionistic fuzzy $\mathcal{T}$ equivalence information systems. At the first, we introduce the concepts of knowledge granulation, knowledge entropy and knowledge uncertainty measure in Atanassov's intuitionistic fuzzy $\mathcal{T}$ equivalence information systems, and some important properties of them are studied. From these properties, it can be shown that these measures provide important approaches to measuring the discernibility ability of different knowledge in Atanassov's intuitionistic fuzzy $\mathcal{T}$ equivalence information systems. And relationships among knowledge granulation, knowledge entropy and knowledge uncertainty measure are considered. Furthermore, we introduce the definition of rough entropy of rough sets in Atanassov's intuitionistic fuzzy $\mathcal{T}$ equivalence information systems. By an example, it is shown that the rough entropy of rough set is more accurate than natural extension of classical rough degree to measure the roughness of rough set in Atanassov's intuitionistic fuzzy $\mathcal{T}$ equivalence information systems.

Keywords: Information system, Atanassov's intuitionistic fuzzy $\mathcal{T}$ equivalence relation, Atanassov's intuitionistic fuzzy rough sets, uncertainty measure

1. Introduction

The theory of rough sets, proposed by Pawlak [19, 20], is a powerful mathematical approach to deal with inexact, uncertain or vague knowledge. It has been successfully applied to various fields of artificial intelligence such as pattern recognition, machine learning, and automated knowledge acquisition. In recent years, The generalization of classical rough set model is one of the most important study spotlights.

It is widely acknowledge that classical Pawlak rough set theory is based on an assumption that every object in the universe of discourse is associated with some information. In many practical issues, it may happen that some of the attribute values for an object are a pair of fuzzy-valued. For this reason, in 1986, Atanassov [1] proposed the concept of an Atanassov's intuitionistic fuzzy (IF) set, which is very effective to deal with vagueness. The concept of IF set is a generalization of the fuzzy set [37] defined by a pair of membership functions which is a membership degree and a non-membership degree. The membership and non-membership values induce an indeterminacy index, which models the hesitancy of deciding the degree to which an object satisfies a particular property. As a generalization of the fuzzy set, the concept of IF set has played an important role in the analysis of uncertainty of data. Recently, IF set theory has been successfully applied in decision analysis and pattern recognition [3, 8, 9, 11, 13, 18, 28, 31, 35, 36, 39, 40].

Combining IF set theory and rough set theory may result in a new hybrid mathematical structure for the

*Corresponding author. W.H. Xu, School of Mathematics and Statistics, Chongqing University of Technology, Chongqing 400054, P.R. China. Tel.: +86 23 62563057; E-mail: chxuwh@gmail.com.

requirement of knowledge-handling systems. Research on this topic has been investigated by a number of authors. Coker [4] first revealed the relationships between IF set theory and rough set theory and showed that a fuzzy rough set was in fact an Atanassov's intuitionistic L fuzzy set. Various tentative definitions of IF rough sets were explored to extend rough set theory to the IF environment [5, 6, 12, 24–26]. For example, according to fuzzy rough sets in the sense of Nanda and Majumda [17], Jena and Ghosh [12] and Chakrabarty et al. [5] independently proposed the concept of an IF rough set in which the lower and upper approximations are both IF sets.

Entropy is an important concept proposed by Shannon [27] to evaluate uncertainty of a system. It is a very useful mechanism for characterizing information contents in various modes and has been applied in diverse fields. Wei et al. [30] propose an entropy measure for interval-valued Atanassov's intuitionistic fuzzy sets and give an approach to construct similarity measures by entropy measures for interval-valued Atanassov's intuitionistic fuzzy sets. The extension of entropy and its variants were adapted for rough set in [2, 10, 14, 21, 29]. For example, Duentsch and Gediga defined the information entropy and three kinds of conditional entropy in rough sets for predicting a decision attribute [10]. Beaubouef et al. [2] proposed a method measuring uncertainty of rough sets and rough relation databases based on rough entropy. Wierman [29] presented the measures of uncertainty and granularity in rough set theory, along with an axiomatic derivation. Liang et al. [14] proposed a new method for evaluating both uncertainty and fuzziness. Qian and Liang [21] proposed a combination entropy for evaluating uncertainty of a knowledge from an information system. Beaubouef et al. [2] proposed a method measuring uncertainty of rough sets and rough relation databases based on rough entropy. All these studies were dedicated to evaluating uncertainty of a set in terms of the partition ability of a knowledge. As a powerful mechanism, granulation was introduced by Zadeh [38]. It presents a more visual and easily understandable description for a partition on the universe. From the viewpoint of granulation, several measures on knowledge in an information system were proposed and the relationships among these measures were discussed by Liang et al. in [15, 16]. These measures include granulation measure, information entropy, rough entropy, and knowledge granulation, and have become effective mechanisms for evaluating uncertainty in rough set theory. Qian et al. studied knowledge granulation in a knowledge base [22], and fuzzy information granularity in a binary granular structure [23]. Xu et al. [32] introduced concepts of knowledge granulation, knowledge entropy and knowledge uncertainty measure in ordered information systems.

So far, however, uncertainty measures in IF $\mathcal{T}$ equivalence information systems have not been reported. In this paper, we aim to address uncertainty measure issue in IF $\mathcal{T}$ equivalence information systems. This paper introduces knowledge granulation, knowledge entropy and knowledge uncertainty measure into IF $\mathcal{T}$ equivalence information systems, and discuss some properties of them. It is shown that these proposed measures provide approaches to measuring the discernibility ability of different knowledge in set-valued information systems.

The rest of this article is organized as follows. In Section 2, we recall the basic contents of IF information system and IF rough sets based on IF $\mathcal{T}$ equivalence relation . In Section 3–5, knowledge granulation, knowledge entropy and knowledge uncertainty measure are introduced in IF $\mathcal{T}$ equivalence information systems, and some important properties of them are discussed. In Section 6, we investigate the relationships and differences among knowledge granulation, knowledge entropy and knowledge uncertainty measure. Finally, as an application of knowledge granulation, we introduce definition of rough entropy of rough set in ordered information systems in Section 7. By an example, it is shown that the rough entropy of rough set is more accurate than the natural extension of classical rough degree to measure the roughness of rough set in IF $\mathcal{T}$ equivalence information systems.

2. Atanassov's intuitionistic fuzzy rough sets and Atanassov's intuitionistic fuzzy $\mathcal{T}$ equivalence information systems

In this section, we mainly introduce the basic contents of IF information system and IF rough sets based on IF $\mathcal{T}$ equivalence relation. We omit review of traditional rough set theory and detailed description of the rough set theory be found in the source papers [19].

Definition 2.1. [7] Let $L^* = \{(\alpha_1, \alpha_2) \in I^2 | \alpha_1 + \alpha_2 \leq 1\}$. We define a relation $\leq_{L^*}$ on L^* as follows: for all $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in L^*$, $(\alpha_1, \alpha_2) \leq_{L^*} (\beta_1, \beta_2) \Leftrightarrow \alpha_1 \leq \beta_1$ and $\alpha_2 \geq \beta_2$. Then the relation $\leq_{L^*}$ is a partial ordering on L^* and the pair $(L^*, \leq_{L^*})$ is a complete lattice with the smallest element $0_{L^*} = (0, 1)$

and the greatest element $1_{L^*} = (1, 0)$. The meet operator $\wedge$, join operator $\vee$ and complement operator $\sim$ on $(L^*, \leq_{L^*})$ which are linked to the ordering $\leq_{L^*}$ are, respectively, defined as follows: for all $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in L^*$,

$(\alpha_1, \alpha_2) \wedge (\beta_1, \beta_2) = (\min(\alpha_1, \beta_1), \max(\alpha_2, \beta_2))$,

$(\alpha_1, \alpha_2) \vee (\beta_1, \beta_2) = (\max(\alpha_1, \beta_1), \min(\alpha_2, \beta_2))$.

$\sim (\alpha_1, \alpha_2) = (\alpha_2, \alpha_1)$.

Meanwhile we introduce an order relation $\geq_{L^*}$ on L^* as follows: for all $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in L^*$,

$(\beta_1, \beta_2) \geq_{L^*} (\alpha_1, \alpha_2) \Leftrightarrow (\alpha_1, \alpha_2) \leq_{L^*} (\beta_1, \beta_2)$,

$\alpha = \beta \Leftrightarrow \alpha \leq_{L^*} \beta$ and $\beta \leq_{L^*} \alpha \Leftrightarrow \alpha_1 = \beta_1, \alpha_2 = \beta_2$,

$\alpha <_{L^*} \beta \Leftrightarrow \alpha \leq_{L^*} \beta$ and $\alpha \neq \beta$.

Definition 2.2. [1] Let a set U be fixed. An IF set $\widetilde{A}$ in U is an object having the form

$$\widetilde{A} = \{\langle x, \mu_{\widetilde{A}}(x), \nu_{\widetilde{A}}(x)\rangle | x \in U\},$$

where $\mu_{\widetilde{A}} : U \to I$ and $\nu_{\widetilde{A}} : U \to I$ satisfy $0 \leq \mu_{\widetilde{A}}(x) + \nu_{\widetilde{A}}(x) \leq 1$ for all $x \in U$, $\mu_{\widetilde{A}}(x)$ and $\nu_{\widetilde{A}}(x)$ are called the degree of membership and the degree of non-membership of the element $x \in U$ to $\widetilde{A}$, respectively. The family of all IF subsets of U is denoted by $IF(U)$. The complement of an IF set $\widetilde{A}$ is defined by $\sim \widetilde{A} = \{\langle x, \nu_{\widetilde{A}}(x), \mu_{\widetilde{A}}(x)\rangle | x \in U\}$.

Obviously, every fuzzy set $\widetilde{A} = \{\langle x, \mu_{\widetilde{A}}(x)\rangle | x \in U\}$ can be identified with the IF set of the form $\widetilde{A} = \{\langle x, \mu_{\widetilde{A}}(x), 1 - \mu_{\widetilde{A}}(x)\rangle | x \in U\}$. We denote the family of all fuzzy subsets of U as $F(U)$.

Definition 2.3. [1] If $\widetilde{A}, \widetilde{B} \in IF(U)$, then,

(1) $\widetilde{A} \subseteq \widetilde{B} \Leftrightarrow \mu_{\widetilde{A}}(x) \leq \mu_{\widetilde{B}}(x)$ and $\nu_{\widetilde{A}}(x) \geq \nu_{\widetilde{B}}(x)$ for all $x \in U$,
(2) $\widetilde{A} \supseteq \widetilde{B} \Leftrightarrow \widetilde{B} \subseteq \widetilde{A}$,
(3) $\widetilde{A} = \widetilde{B} \Leftrightarrow \widetilde{A} \subseteq \widetilde{B}$ and $\widetilde{B} \subseteq \widetilde{A}$,

Definition 2.4. [1] If $\widetilde{A}, \widetilde{B} \in IF(U)$, then,

(1) $\widetilde{A} \cap \widetilde{B} = \{\langle x, \min(\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(x)), \max(\nu_{\widetilde{A}}(x), \nu_{\widetilde{B}}(x))| x \in U\rangle\}$,
(2) $\widetilde{A} \cup \widetilde{B} = \{\langle x, \max(\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(x)), \min(\nu_{\widetilde{A}}(x), \nu_{\widetilde{B}}(x))| x \in U\rangle\}$.

For $\alpha = (\alpha_1, \alpha_2) \in L^*$, $\widetilde{\alpha} = \widetilde{(\alpha_1, \alpha_2)}$ will be denoted by the constant IF set: $\widetilde{\alpha}(x) = \widetilde{(\alpha_1, \alpha_2)}(x) = (\alpha_1, \alpha_2)$, for all $x \in U$. In particularly, if $a \in I$ we denote $\widetilde{a}$ as a constant fuzzy set, i.e., $\widetilde{a}(x) = a$ for all $x \in U$.

For any $y \in U$, IF set $\widetilde{1}_y$ and $\widetilde{1}_{U-\{y\}}$ are, respectively, define as follows: for $x \in U$,

$$\mu_{\widetilde{1}_{\{y\}}}(x) = \begin{cases} 1, \text{ if } x = y, \\ 0. \text{ if } x \neq y. \end{cases} \quad \nu_{\widetilde{1}_{\{y\}}}(x) = \begin{cases} 0, \text{ if } x = y, \\ 1. \text{ if } x \neq y. \end{cases}$$

$$\mu_{\widetilde{1}_{U-\{y\}}}(x) = \begin{cases} 0, \text{ if } x = y, \\ 1. \text{ if } x \neq y. \end{cases} \quad \nu_{\widetilde{1}_{U-\{y\}}}(x) = \begin{cases} 1, \text{ if } x = y, \\ 0. \text{ if } x \neq y. \end{cases}$$

The IF universe set is $\widetilde{1}_U = \widetilde{(1, 0)} = \widetilde{1}_{L^*} = \{\langle x, 1, 0\rangle | x \in U\}$ and the IF empty set is $\widetilde{1}_\emptyset = \widetilde{(0, 1)} = \widetilde{0}_{L^*} = \{\langle x, 0, 1\rangle | x \in U\}$.

Definition 2.5. [1] A fuzzy triangular norm (briefly fuzzy t-norm) on I is an increasing, commutative, associative mapping $T : I \times I \to I$ satisfying $T(1, a) = a$ for all $a \in I$.

A fuzzy triangular t-conorm (briefly fuzzy t-conorm) on I is an increasing, commutative, associative mapping $S : I \times I \to I$ satisfying $S(0, a) = a$ for all $a \in I$.

A fuzzy t-norm T and a fuzzy t-conorm S on I are said to be dual with respect to complement operator $\sim$, if for all $a, b \in I$,

$S(a, b) =\sim T(1 - a, 1 - b) = 1 - T(1 - a, 1 - b)$.

Definition 2.6. [7] An IF t-norm $\mathcal{T}$(respectively, t-conorm $\mathcal{S}$) on L^* can be defined by fuzzy t-norm T (t-conorm S) as follows:

$\mathcal{T}(\alpha, \beta) = (T(\alpha_1, \beta_1), S(\alpha_2, \beta_2))$,

$\mathcal{S}(\alpha, \beta) = (S^{'}(\alpha_1, \beta_1), T^{'}(\alpha_2, \beta_2))$,

for all $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in L^*$

Definition 2.7. Let T be a fuzzy t-norm on I and S the dual of T. Two fuzzy residual implication by the T and S can be defined as follows:

$\theta(a, b) = \sup\{c \in I | T(a, c) \leq b\}$, $\phi(a, b) = \inf\{c \in I | S(a, c) \geq b\}$,

for any $a, b, c \in I$.

Now, we define the following two IF implication on L^*: for all $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in L^*$,

$\Phi(\alpha, \beta) = (\phi(1 - \alpha_2, \beta_1), \theta(1 - \alpha_1, \beta_2))$,

$\Theta(\alpha, \beta) = (\theta(1 - \alpha_2, \beta_1), \phi(1 - \alpha_1, \beta_2))$.

Proposition 2.1. *Let θ be a fuzzy residual implication and ϕ be the dual of θ, for any $a, b \in I$, then $\phi(\sim a, \sim b) =\sim \theta(a, b)$.*

Obviously, it can be seen that $\Phi(\alpha, \beta) =\sim \Theta(\sim \alpha, \sim \beta)$, for all $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2) \in L^*$.

Definition 2.8. [3] An IF binary relation $\widetilde{R}$ on U is an IF subset of $U \times U$, namely, $\widetilde{R}$ is given by

$$\widetilde{R} = \{\langle (x, y), \mu_{\widetilde{R}}(x, y), \nu_{\widetilde{R}}(x, y)\rangle | (x, y) \in U \times U\},$$

where $\mu_{\widetilde{R}} : U \times U \to I$ and $\nu_{\widetilde{R}} : U \times U \to I$, $0 \leq \mu_{\widetilde{R}}(x, y) + \nu_{\widetilde{R}}(x, y) \leq 1$ for all $(x, y) \in U \times U$. $IFR(U \times U)$ will be used to denote the family of all IF relations on U.

Definition 2.9. [3] Let $\widetilde{R} \in IFR(U \times U)$, we say that

(1) $\widetilde{R}$ is referred to as an IF reflexive relation if for any $x \in U$, $\widetilde{R}(x, x) = (1, 0)$.
(2) $\widetilde{R}$ is referred to as an IF symmetric relation if for any $x, y \in U$, $\widetilde{R}(x, y) = \widetilde{R}(y, x)$.
(3) $\widetilde{R}$ is referred to as an IF $\mathcal{T}$ transitive relation if for any $x, y, z \in U$, $\widetilde{R}(x, z) \geq_{L^*} \mathcal{T}(\widetilde{R}(x, y), \widetilde{R}(y, z))$.

If $\widetilde{R}$ is IF reflexive, IF symmetric and IF $\mathcal{T}$ transitive on U, then we say that $\widetilde{R}$ is an IF $\mathcal{T}$ equivalence relation on U.

An IF $\mathcal{T}$ equivalence class $[x_i]_{\widetilde{R}}$ of $x_i \in U$ is an IF set, denoted as: $[x_i]_{\widetilde{R}} = (\mu_{[x_i]_{\widetilde{R}}}, \nu_{[x_i]_{\widetilde{R}}})$, where $\mu_{[x_i]_{\widetilde{R}}} = (\mu_{\widetilde{R}(i,1)}, \mu_{\widetilde{R}(i,2)}, \ldots, \mu_{\widetilde{R}(i,n)})$, $\mu_{\widetilde{R}(i,j)} = \mu_{\widetilde{R}}(x_i, x_j)$ and $\nu_{[x_i]_{\widetilde{R}}} = (\nu_{\widetilde{R}(i,1)}, \nu_{\widetilde{R}(i,2)}, \ldots, +\nu_{\widetilde{R}(i,n)})$, $\nu_{\widetilde{R}(i,j)} = \nu_{\widetilde{R}}(x_i, x_j)$. The IF $\mathcal{T}$ equivalence class is an IF knowledge granule, the elements in the class are IF $\mathcal{T}$ equivalent indiscernible. The family of the IF equivalence classes $[x_i]_{\widetilde{R}}$, written as $U/\widetilde{R} = \{[x_i]_{\widetilde{R}} | x_i \in U\}$ is called an IF quotient set (or classification) of U induced by $\widetilde{R}$.

There are two kinds of special case of classification $U/\widetilde{R}$, i.e., the discrete case and the indiscrete case. The discrete case is defined as:

$$U/\widetilde{I_R} : U/\widetilde{I_R} = \{[x_i]_{\widetilde{I_R}} = \widetilde{1}_{\{x_i\}} | x_i \in U\}.$$

The indiscrete case is defined as:

$$U/\widetilde{\delta_R} : U/\widetilde{\delta_R} = \{[x_i]_{\widetilde{\delta_R}} = \widetilde{1}_U | x_i \in U\}.$$

An IF information system [33, 34] is an ordered quadruple $I = (U, AT, V, f)$, where $U = \{x_1, x_2, \ldots, x_n\}$ is a non-empty finite set of objects, $AT = \{a_1, a_2, \ldots, a_p\}$ is a non-empty finite set of attributes, $V = \bigcup_{a \in AT} V_a$ and V_a is a domain of attribute a, $f : U \times AT \to V$ is a function such that $f(x, a) \in V_a$, for each $a \in AT$, $x \in U$, called an information function, where V_a is an IF set of the universe U. That is

$$f(x, a) = (\mu_a(x), \nu_a(x)), \text{ for all } a \in AT,$$

Table 1
An IF information system

U	a_1	a_2	a_3	a_4	a_5
x_1	(0.4, 0.6)	(0.8, 0.1)	(0.6, 0.3)	(0.9, 0.0)	(0.7, 0.1)
x_2	(0.3, 0.5)	(0.7, 0.3)	(0.5, 0.1)	(0.7, 0.1)	(0.6, 0.3)
x_3	(0.5, 0.3)	(0.8, 0.1)	(0.7, 0.1)	(1.0, 0.0)	(0.7, 0.1)
x_4	(0.6, 0.3)	(0.9, 0.0)	(0.7, 0.1)	(0.8, 0.2)	(0.8, 0.0)
x_5	(0.9, 0.1)	(0.9, 0.0)	(0.8, 0.1)	(0.6, 0.3)	(1.0, 0.0)

where $\mu_a : U \to [0, 1]$ and $\nu_a : U \to [0, 1]$ satisfy $0 \leq \mu_a(x) + \nu_a(x) \leq 1$, for all $x \in U$. μ_a and ν_a are, respectively, called the degree of membership and the degree of non-membership of the element $x \in U$ to attribute a. We denote $\widetilde{a}(x) = (\mu_a(x), \nu_a(x))$, then it is clear that $\widetilde{a}$ is an IF set of U. Table 1 shows an IF information system.

Definition 2.10. An IF $\mathcal{T}$ equivalence information system is an ordered quintuple $\widetilde{\mathcal{I}} = (U, AT, V, f, \digamma)$, where (U, AT, V, f) is an IF information system, $\digamma$ is the mapping from power set AT into the family set $\widetilde{\mathbf{R}}$ of IF $\mathcal{T}$ equivalence relation.

Let $\widetilde{\mathcal{I}} = (U, AT, V, f, \digamma)$ be an IF $\mathcal{T}$ equivalence information system, for $A \subseteq AT$, $a \in A$, $\widetilde{R}_a \in \widetilde{\mathbf{R}}$ be an IF $\mathcal{T}$ equivalence relation respect to attribute a. Denotes $\widetilde{R}_A = \bigcap_{a \in A} \widetilde{R}_a$.

For simplicity, the examples throughout the paper will exploit the relation $\widetilde{R}_a$ as following: $\widetilde{R}_a(x_i, x_j) = (\mu_{\widetilde{R}_a}(x_i, x_j), \nu_{\widetilde{R}_a}(x_i, x_j))$, where, $\mu_{\widetilde{R}_a}(x_i, x_j) = 1 - \max\{|\mu_a(x_i) - \mu_a(x_j)|, |\nu_a(x_i) - \nu_a(x_j)|\}$ and $\nu_{R_a}(x_i, x_j) = \frac{1}{2}(|\mu_a(x_i) - \mu_a(x_j)| + |\nu_a(x_i) - \nu_a(x_j)|)$. Consider the IF t-norm $\mathcal{T}$: $\mathcal{T}(\alpha, \beta) = (\max\{0, \alpha_1 + \beta_1 - 1\}, \min\{1, \alpha_2 + \beta_2\})$ for $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in L^*$. Obviously, the relation $\widetilde{R}_a$ is an IF $\mathcal{T}$ equivalence relation.

Definition 2.11. Let $\widetilde{\mathcal{I}} = (U, AT, V, f, \digamma)$ be an IF $\mathcal{T}$ equivalence information system. $A, B \subseteq AT$, and $U/\widetilde{R}_A = \{[x_i]_{\widetilde{R}_A} | x_i \in U\}$, $U/\widetilde{R}_B = \{[x_i]_{\widetilde{R}_B} | x_i \in U\}$ be classification of two IF $\mathcal{T}$ equivalence relations $\widetilde{R}_A$ and $\widetilde{R}_B$ respectively.

(1) If $[x_i]_{\widetilde{R}_A} = [x_i]_{\widetilde{R}_B}$ for all $x_i \in U$, then we call that classification $U/\widetilde{R}_A$ is equal to $U/\widetilde{R}_B$, denoted by $U/\widetilde{R}_A = U/\widetilde{R}_B$.
(2) If $[x_i]_{\widetilde{R}_A} \subseteq [x_i]_{\widetilde{R}_B}$ for all $x_i \in U$, then we call that classification $U/\widetilde{R}_A$ is finer than $U/\widetilde{R}_B$, denoted by $U/\widetilde{R}_A \subseteq U/\widetilde{R}_B$.
(3) If $[x_i]_{\widetilde{R}_A} \subseteq [x_i]_{\widetilde{R}_B}$ for all $x_i \in U$ and $[x_i]_{\widetilde{R}_A} \subset [x_i]_{\widetilde{R}_B}$ for some $x_i \in U$, then we call that classifi-

cation $U/\widetilde{R}_A$ is properly finer than $U/\widetilde{R}_B$, denoted by $U/\widetilde{R}_A \subset U/\widetilde{R}_B$.

We denote $\widetilde{R}_A = \widetilde{R}_B \Leftrightarrow U/\widetilde{R}_A = U/\widetilde{R}_B$, $\widetilde{R}_A \preceq \widetilde{R}_B \Leftrightarrow U/\widetilde{R}_A \subseteq U/\widetilde{R}_B$ and $\widetilde{R}_A \prec \widetilde{R}_B \Leftrightarrow U/\widetilde{R}_A \subset U/\widetilde{R}_B$.

Example 2.1. From Table 1, the IF $\mathcal{T}$ equivalence relation $\widetilde{R}_{AT}$ is computed as follows:

$$\widetilde{R}_{AT} = \begin{pmatrix} (1.0, 0) & (0.8, 0.15) & (0.7, 0.2) & (0.7, 0.25) & (0.5, 0.5) \\ (0.8, 0.15) & (1.0, 0) & (0.7, 0.2) & (0.7, 0.25) & (0.4, 0.5) \\ (0.7, 0.2) & (0.7, 0.2) & (1.0, 0) & (0.8, 0.2) & (0.6, 0.35) \\ (0.7, 0.25) & (0.7, 0.25) & (0.8, 0.2) & (1.0, 0) & (0.7, 0.25) \\ (0.5, 0.5) & (0.4, 0.5) & (0.6, 0.35) & (0.7, 0.25) & (1.0, 0) \end{pmatrix}$$

If denote $A = \{a_1, a_2, a_3\}$, the IF $\mathcal{T}$ equivalence relation $\widetilde{R}_A$ is computed as follows:

$$\widetilde{R}_{A} = \begin{pmatrix} (1.0, 0) & (0.8, 0.15) & (0.7, 0.2) & (0.7, 0.25) & (0.5, 0.5) \\ (0.8, 0.15) & (1.0, 0) & (0.8, 0.2) & (0.7, 0.25) & (0.4, 0.5) \\ (0.7, 0.2) & (0.8, 0.2) & (1.0, 0) & (0.9, 0.1) & (0.6, 0.3) \\ (0.7, 0.25) & (0.7, 0.25) & (0.9, 0.1) & (1.0, 0) & (0.7, 0.25) \\ (0.5, 0.5) & (0.4, 0.5) & (0.6, 0.3) & (0.7, 0.25) & (1.0, 0) \end{pmatrix}$$

Thus, it is obvious that $U/\widetilde{R}_{AT} \subseteq U/\widetilde{R}_A$. We can say that classification $U/\widetilde{R}_{AT}$ is finer than classification $U/\widetilde{R}_A$, or knowledge $\widetilde{R}_{AT}$ is finer than $\widetilde{R}_A$.

Let $\widetilde{\mathcal{I}} = (U, AT, V, f, \digamma)$ be an IF $\mathcal{T}$ equivalence information system. $\widetilde{X} \in IF(U)$ and $A \subseteq AT$, the Φ-upper and Θ-lower approximations of $\widetilde{X}$ with respect to IF relation $\widetilde{R}_A$ are respectively defined by

$\overline{\widetilde{R}_A}(\widetilde{X}) = \{\langle x, \mu_{\overline{\widetilde{R}_A}(\widetilde{X})}(x), \nu_{\overline{\widetilde{R}_A}(\widetilde{X})}(x)\rangle | x \in U\}$;

$\underline{\widetilde{R}_A}(\widetilde{X}) = \{\langle x, \mu_{\underline{\widetilde{R}_A}(\widetilde{X})}(x), \nu_{\underline{\widetilde{R}_A}(\widetilde{X})}(x)\rangle | x \in U\}$.

where

$\mu_{\overline{\widetilde{R}_A}(\widetilde{X})}(x) = \bigvee_{y\in U} \phi(1 - \mu_{\widetilde{R}_A}(x, y), \mu_{\widetilde{X}}(y))$,

$\nu_{\overline{\widetilde{R}_A}(\widetilde{R})}(x) = \bigwedge_{y\in U} \theta(1 - \nu_{\widetilde{R}_A}(x, y), \nu_{\widetilde{X}}(y))$;

$\mu_{\underline{\widetilde{R}_A}(\widetilde{X})}(x) = \bigwedge_{y\in U} \theta(1 - \nu_{\widetilde{R}_A}(x, y), \mu_{\widetilde{X}}(y))$,

$\nu_{\underline{\widetilde{R}_A}(\widetilde{X})}(x) = \bigvee_{y\in U} \phi(1 - \mu_{\widetilde{R}_A}(x, y), \nu_{\widetilde{X}}(y))$.

From the above definition of IF rough approximation, the following important properties in $\mathcal{T}$ equivalence information systems have been proved.

Proposition 2.2. *Let $\widetilde{\mathcal{I}} = (U, AT, V, f, \digamma)$ be an IF $\mathcal{T}$ equivalence information system. $\widetilde{X}, \widetilde{Y} \in IF(U)$, $\alpha = (\alpha_1, \alpha_2)$ then its Φ-upper and Θ-lower approximations satisfy the following properties.*

(1) $\underline{\widetilde{R}_A}(\sim \widetilde{X}) = \sim\overline{\widetilde{R}_A}(\widetilde{X})$, $\overline{\widetilde{R}_A}(\sim \widetilde{X}) = \sim \underline{\widetilde{R}_A}(\widetilde{X})$.

(2) $\underline{\widetilde{R}_A}(\widetilde{X}) \subseteq \widetilde{X} \subseteq \overline{\widetilde{R}_A}(\widetilde{X})$.

(3) $\underline{\widetilde{R}_A}(\widetilde{X} \cap \widetilde{Y}) = \underline{\widetilde{R}_A}(\widetilde{X}) \cap \underline{\widetilde{R}_A}(\widetilde{Y})$, $\overline{\widetilde{R}_A}(\widetilde{X} \cup \widetilde{Y}) = \overline{\widetilde{R}_A}(\widetilde{X}) \cup \overline{\widetilde{R}_A}(\widetilde{Y})$.

(4) $\widetilde{X} \subseteq \widetilde{Y} \Rightarrow \underline{\widetilde{R}_A}(\widetilde{X}) \subseteq \underline{\widetilde{R}_A}(\widetilde{Y})$ and $\overline{\widetilde{R}_A}(\widetilde{X}) \subseteq \overline{\widetilde{R}_A}(\widetilde{Y})$.

(5) $\underline{\widetilde{R}_A}(\widetilde{X} \cup \widetilde{Y}) \supseteq \underline{\widetilde{R}_A}(\widetilde{X}) \cup \underline{\widetilde{R}_A}(\widetilde{Y})$, $\overline{\widetilde{R}_A}(\widetilde{X} \cap \widetilde{Y}) \subseteq \overline{\widetilde{R}_A}(\widetilde{X}) \cup \overline{\widetilde{R}_A}(\widetilde{Y})$.

(6) $\underline{\widetilde{R}_A}(\widetilde{\alpha}) = \widetilde{\alpha}$, $\overline{\widetilde{R}_A}(\widetilde{\alpha}) = \widetilde{\alpha}$.

In particular, $\underline{\widetilde{R}_A}(\widetilde{1}_\emptyset) = \overline{\widetilde{R}_A}(\widetilde{1}_\emptyset) = \widetilde{1}_\emptyset$, $\underline{\widetilde{R}_A}(\widetilde{1}_U) = \overline{\widetilde{R}_A}(\widetilde{1}_U) = \widetilde{1}_U$.

(7) $\underline{\widetilde{R}_A}(\underline{\widetilde{R}_A}(\widetilde{X})) = \underline{\widetilde{R}_A}(\widetilde{X})$, $\overline{\widetilde{R}_A}(\overline{\widetilde{R}_A}(\widetilde{X})) = \overline{\widetilde{R}_A}(\widetilde{X})$.

3. Knowledge granulation in IF $\mathcal{T}$ equivalence information systems

In this section, we will introduce the definition of granulation of knowledge in $\mathcal{T}$ equivalence information systems, and discuss some important properties.

Definition 3.1. Let $\widetilde{\mathcal{I}} = (U, AT, V, f, \digamma)$ be an IF $\mathcal{T}$ equivalence information system, $A \subseteq AT$, $\widetilde{R}_A$ be an IF $\mathcal{T}$ equivalence relation, $U/\widetilde{R}_A = \{[x_i]_{\widetilde{R}_A} | x_i \in U\}$ be the classification. And we can define the IF cardinality of $[x_i]_{\widetilde{R}_A}$ as following:

$$\widetilde{\mathrm{Card}}|[x_i]_{\widetilde{R}_A}| = (|\mu_{[x_i]_{\widetilde{R}_A}}|, |\nu_{[x_i]_{\widetilde{R}_A}}|),$$

where $|\mu_{[x_i]_{\widetilde{R}_A}}| = \sum_{j=1}^{n} \mu_{\widetilde{R}_A(i,j)}$ and $|\nu_{[x_i]_{\widetilde{R}_A}}| = \sum_{j=1}^{n} \nu_{\widetilde{R}_A(i,j)}$.

Definition 3.2. Let $\widetilde{\mathcal{I}} = (U, AT, V, f, F)$ be an IF $\mathcal{T}$ equivalence information system, $A \subseteq AT$, $\widetilde{R}_A$ be an IF $\mathcal{T}$ equivalence relation, and $U/\widetilde{R}_A = \{[x_i]_{\widetilde{R}_A} | x_i \in U\}$ be the classification. Granulation of knowledge $\widetilde{R}$ is defined as

$$\widetilde{GK}(\widetilde{R}_A) = \frac{1}{2|U|^2} \sum_{i=1}^{|U|} (|\mu_{[x_i]_{\widetilde{R}_A}}| + |1 - \nu_{[x_i]_{\widetilde{R}_A}}|).$$

Theorem 3.1. *Let $\widetilde{\mathcal{I}} = (U, AT, V, f, F)$ be an IF $\mathcal{T}$ equivalence information system, $A, B \subseteq AT$, and $U/\widetilde{R}_A = \{[x_i]_{\widetilde{R}_A} | x_i \in U\}$, $U/\widetilde{R}_B = \{[x_i]_{\widetilde{R}_B} | x_i \in U\}$ be classification of two IF $\mathcal{T}$ equivalence relations $\widetilde{R}_A$ and $\widetilde{R}_B$, respectively. If exists a bijective map $h : U/\widetilde{R}_A \to U/\widetilde{R}_B$, such that $|[x_i]_{\widetilde{R}_A}| = |h([x_i]_{\widetilde{R}_A})|$, then $\widetilde{GK}(\widetilde{R}_A) = \widetilde{GK}(\widetilde{R}_B)$.*

Proof. It can be achieved by Definition 3.1. □

Corollary 3.1. Let $\widetilde{\mathcal{I}} = (U, AT, V, f, F)$ be an IF $\mathcal{T}$ equivalence information system, $A, B \subseteq AT$, $\widetilde{R}_A$ and $\widetilde{R}_B$ be two IF $\mathcal{T}$ equivalence relation. If $\widetilde{R}_A = \widetilde{R}_B$, then $\widetilde{GK}(\widetilde{R}_A) = \widetilde{GK}(\widetilde{R}_B)$.

Theorem 3.2. *Let $\widetilde{\mathcal{I}} = (U, AT, V, f, F)$ be an IF $\mathcal{T}$ equivalence information system, $A, B \subseteq AT$, and $U/\widetilde{R}_A = \{[x_i]_{\widetilde{R}_A} | x_i \in U\}$, $U/\widetilde{R}_B = \{[x_i]_{\widetilde{R}_B} | x_i \in U\}$ be classification of two IF $\mathcal{T}$ equivalence relations $\widetilde{R}_A$ and $\widetilde{R}_B$, respectively. If $\widetilde{R}_A \preceq \widetilde{R}_B$, then $\widetilde{GK}(\widetilde{R}_A) \leq \widetilde{GK}(\widetilde{R}_B)$.*

Proof. Since $\widetilde{R}_A \preceq \widetilde{R}_B$, we can have that for all $x_i \in U, [x_i]_{\widetilde{R}_A} \subseteq [x_i]_{\widetilde{R}_B}$. So $|\mu_{[x_i]_{\widetilde{R}_A}}| + |1 - \nu_{[x_i]_{\widetilde{R}_A}}| \leq |\mu_{[x_i]_{\widetilde{R}_B}}| + |1 - \nu_{[x_i]_{\widetilde{R}_B}}|$. Thus, the following holds. i.e.,

$$\begin{aligned}\widetilde{GK}(\widetilde{R}_A) &= \frac{1}{2|U|^2} \sum_{i=1}^{|U|} (|\mu_{[x_i]_{\widetilde{R}_A}}| + |1 - \nu_{[x_i]_{\widetilde{R}_A}}|) \\ &\leq \frac{1}{2|U|^2} \sum_{i=1}^{|U|} (|\mu_{[x_i]_{\widetilde{R}_B}}| + |1 - \nu_{[x_i]_{\widetilde{R}_B}}|) \\ &= \widetilde{GK}(\widetilde{R}_B).\end{aligned}$$

□

Example 3.1. (Continued from Example 2.1) By computing, we have that

$$\begin{aligned}\widetilde{GK}(\widetilde{R}_{AT}) = \frac{1}{2 \times 5^2}(7.6 + 7.5 \\ + 7.85 + 7.95 + 6.6) = 0.75,\end{aligned}$$

$$\begin{aligned}\widetilde{GK}(\widetilde{R}_A) = \frac{1}{2 \times 10^2}(7.6 + 7.6 \\ + 8.2 + 8.15 + 6.65) = 0.764,\end{aligned}$$

Obviously, $\widetilde{GK}(\widetilde{R}_{AT}) \leq \widetilde{GK}(\widetilde{R}_A)$.

Corollary 3.2. Let $\widetilde{\mathcal{I}} = (U, AT, V, f, F)$ be an IF $\mathcal{T}$ equivalence information system, $A, B \subseteq AT$, and $\widetilde{R}_A$, $\widetilde{R}_B$ be two IF $\mathcal{T}$ equivalence relations. If $\widetilde{R}_A \prec \widetilde{R}_B$, then $\widetilde{GK}(\widetilde{R}_A) < \widetilde{GK}(\widetilde{R}_B)$.

Corollary 3.3. Let $\widetilde{\mathcal{I}} = (U, AT, V, f, F)$ be an IF $\mathcal{T}$ equivalence information system, $A, B \subseteq AT$, and $\widetilde{R}_A$, $\widetilde{R}_B$ be two IF $\mathcal{T}$ equivalence relations. If $\widetilde{R}_A \preceq \widetilde{R}_B$ and $\widetilde{GK}(\widetilde{R}_A) = \widetilde{GK}(\widetilde{R}_B)$, then $\widetilde{R}_A = \widetilde{R}_B$.

Theorem 3.3. *Let $\widetilde{\mathcal{I}} = (U, AT, V, f, F)$ be an IF $\mathcal{T}$ equivalence information system. $A \subseteq AT$, and $\widetilde{R}_A$ be an IF $\mathcal{T}$ equivalence relation. The minimum of knowledge granulation of $\widetilde{\mathcal{I}}$ is $1/|U|$. This value is achieved if and only if $\widetilde{R}_A = \widetilde{I}_R$.*

Proof. Since $U/\widetilde{I}_R = \{[x_i]_{\widetilde{I}_R} = \widetilde{1}_{\{x_i\}} | x_i \in U\}$. So we have

$$\begin{aligned}\widetilde{GK}(\widetilde{I}_R) &= \frac{1}{2|U|^2} \sum_{i=1}^{|U|} (|\mu_{[x_i]_{\widetilde{I}_R}}| + |1 - \nu_{[x_i]_{\widetilde{I}_R}}|) \\ &= \frac{1}{2|U|^2} \sum_{i=1}^{|U|} (1 + 1) = \frac{1}{|U|^2}.\end{aligned}$$

Thus, $\widetilde{GK}(\widetilde{I}_R) = \frac{1}{|U|^2}$. □

Theorem 3.4. *Let $\widetilde{\mathcal{I}} = (U, AT, V, f, F)$ be an IF $\mathcal{T}$ equivalence information system. $A \subseteq AT$, and $\widetilde{R}_A$ be an IF $\mathcal{T}$ equivalence relation. The maximum of knowledge granulation of $\widetilde{\mathcal{I}}$ is 1. This value is achieved if and only if $\widetilde{R}_A = \widetilde{\delta}_R$.*

Proof. Since $U/\widetilde{\delta}_R = \{[x_i]_{\widetilde{\delta}_R} = \widetilde{1}_U | x_i \in U\}$. So we have

$$\begin{aligned}\widetilde{GK}(\widetilde{\delta}_R) &= \frac{1}{2|U|^2} \sum_{i=1}^{|U|} (|\mu_{[x_i]_{\widetilde{\delta}_R}}| + |1 - \nu_{[x_i]_{\widetilde{\delta}_R}}|) \\ &= \frac{1}{2|U|^2} \sum_{i=1}^{|U|} (|U| + |U|) = 1.\end{aligned}$$

Thus, $\widetilde{GK}(\widetilde{I}_R) = 1$. □

Theorem 3.5. *Let $\widetilde{\mathcal{I}} = (U, AT, V, f, \digamma)$ be an IF $\mathcal{T}$ equivalence information system. $A \subseteq AT$, and $\widetilde{R}_A$ be an IF $\mathcal{T}$ equivalence relation.The knowledge granulation $\widetilde{GK}(\widetilde{R}_A)$ exists the boundedness, i.e., $1/|U| \leq \widetilde{GK}(\widetilde{R}_A) \leq 1$. Where $\widetilde{GK}(\widetilde{R}_A) = 1/|U|$ if and only if $\widetilde{R}_A = \widetilde{I}_R$, and $\widetilde{GK}(\widetilde{R}_A) = 1$ if and only if $\widetilde{R}_A = \widetilde{\delta}_R$.*

Proof. It can be obtained by Theorems 3.3 and 3.4. □

Theorem 3.6. *Let $\widetilde{\mathcal{I}} = (U, AT, V, f, \digamma)$ be an IF $\mathcal{T}$ equivalence information system. $A \subseteq AT$, and $U/\widetilde{R}_A = \{[x_i]_{\widetilde{R}_A} | x_i \in U\}$ be classification of IF $\mathcal{T}$ equivalence relation $\widetilde{R}_A$. If some knowledge granule $[x_i]_{\widetilde{R}_A}$ ($x_i \in U$) can be resolved into two new knowledge granules, and else knowledge granule have no change, where we denote the new knowledge by $\widetilde{R'}_A$, then $\widetilde{GK}(\widetilde{R'}_A) \leq \widetilde{GK}(\widetilde{R}_A)$.*

Proof. Assume that $[x_i]_{\widetilde{R}_A}$ of $U/\widetilde{R}_A$ can be resolved into $[x_i]_{\widetilde{R'}_A}$ and $[x_j]_{\widetilde{R'}_A}$ ($i < j$), where $[x_i]_{\widetilde{R}_A} = [x_i]_{\widetilde{R'}_A} \cup [x_j]_{\widetilde{R'}_A}$, and $[x_i]_{\widetilde{R'}_A} \subseteq [x_i]_{\widetilde{R}_A}$, $[x_j]_{\widetilde{R'}_A} \subseteq [x_j]_{\widetilde{R}_A}$. So, we have

$$U/\widetilde{R'}_A = \{[x_1]_{\widetilde{R}_A}, [x_2]_{\widetilde{R}_A}, \cdots, [x_i]_{\widetilde{R'}_A}, \cdots, [x_j]_{\widetilde{R'}_A}, \cdots, [x_{|U|}]_{\widetilde{R}_A}\}.$$

That is to say,

$$\begin{aligned}
\widetilde{GK}(\widetilde{R}_A) &= \frac{1}{2|U|^2}\sum_{t=1}^{|U|}(|\mu_{[x_t]_{\widetilde{R}_A}}| + |1-\nu_{[x_t]_{\widetilde{R}_A}}|)\\
&= \frac{1}{2|U|^2}\sum_{t=1}^{i-1}(|\mu_{[x_t]_{\widetilde{R}_A}}| + |1-\nu_{[x_t]_{\widetilde{R}_A}}|)\\
&\quad + \frac{1}{2|U|^2}(|\mu_{[x_i]_{\widetilde{R}_A}}| + |1-\nu_{[x_i]_{\widetilde{R}_A}}|)\\
&\quad + \frac{1}{2|U|^2}\sum_{t=i+1}^{j-1}(|\mu_{[x_t]_{\widetilde{R}_A}}| + |1-\nu_{[x_t]_{\widetilde{R}_A}}|)\\
&\quad + \frac{1}{2|U|^2}(|\mu_{[x_j]_{\widetilde{R}_A}}| + |1-\nu_{[x_j]_{\widetilde{R}_A}}|)\\
&\quad + \frac{1}{2|U|^2}\sum_{t=j+1}^{|U|}(|\mu_{[x_t]_{\widetilde{R}_A}}| + |1-\nu_{[x_t]_{\widetilde{R}_A}}|)\\
&\geq \frac{1}{2|U|^2}\sum_{t=1}^{i-1}(|\mu_{[x_t]_{\widetilde{R}_A}}| + |1-\nu_{[x_t]_{\widetilde{R}_A}}|)\\
&\quad + \frac{1}{2|U|^2}(|\mu_{[x_i]_{\widetilde{R'}_A}}| + |1-\nu_{[x_i]_{\widetilde{R'}_A}}|)\\
&\quad + \frac{1}{2|U|^2}\sum_{t=i+1}^{j-1}(|\mu_{[x_t]_{\widetilde{R}_A}}| + |1-\nu_{[x_t]_{\widetilde{R}_A}}|)\\
&\quad + \frac{1}{2|U|^2}(|\mu_{[x_j]_{\widetilde{R'}_A}}| + |1-\nu_{[x_j]_{\widetilde{R'}_A}}|)\\
&\quad + \frac{1}{2|U|^2}\sum_{t=j+1}^{|U|}(|\mu_{[x_t]_{\widetilde{R}_A}}| + |1-\nu_{[x_t]_{\widetilde{R}_A}}|)\\
&= \widetilde{GK}(\widetilde{R'}_A)
\end{aligned}$$

Thus, $\widetilde{GK}(\widetilde{R'}_A) \leq \widetilde{GK}(\widetilde{R}_A)$. □

Corollary 3.4. Let $\widetilde{\mathcal{I}} = (U, AT, V, f, \digamma)$ be an IF $\mathcal{T}$ equivalence information system, $A \subseteq AT$, $\widetilde{R}_A$ be an IF $\mathcal{T}$ equivalence relation. If $\widetilde{R}_A$ can be resolved into a new knowledge $\widetilde{R'}_A$, then $\widetilde{GK}(\widetilde{R'}_A) \leq \widetilde{GK}(\widetilde{R}_A)$.

Theorem 3.7. *Let $\widetilde{\mathcal{I}} = (U, AT, V, f, \digamma)$ be an IF $\mathcal{T}$ equivalence information system, $A \subseteq AT$, and $U/\widetilde{R}_A = \{[x_i]_{\widetilde{R}_A} | x_i \in U\}$ be classification of IF $\mathcal{T}$ equivalence relation $\widetilde{R}_A$. If a new knowledge granule can be composed of two knowledge granules of $\widetilde{R}_A$, and else knowledge granules have no change, where we denote the new knowledge by $\widetilde{R''}_A$, then $\widetilde{GK}(\widetilde{R}_A) \leq \widetilde{GK}(\widetilde{R''}_A)$.*

Proof. Assume that $[x_i]_{\widetilde{R''}_A}$ can be composed of $[x_i]_{\widetilde{R}_A}$ and $[x_j]_{\widetilde{R}_A}$ of $U/\widetilde{R}_A$ ($i.j < k$), where $[x_k]_{\widetilde{R''}_A} = [x_i]_{\widetilde{R}_A} \cup [x_j]_{\widetilde{R}_A}$, and $[x_k]_{\widetilde{R}_A} \subseteq [x_k]_{\widetilde{R''}_A}$. So, we have $U/\widetilde{R''}_A = \{[x_1]_{\widetilde{R}_A}, [x_2]_{\widetilde{R}_A}, \ldots, [x_i]_{\widetilde{R}_A}, \ldots, [x_j]_{\widetilde{R}_A}, \ldots, [x_k]_{\widetilde{R''}_A}, \ldots, [x_{|U|}]_{\widetilde{R}_A}\}$. That is to say,

$$\begin{aligned}
\widetilde{GK}(\widetilde{R}_A) &= \frac{1}{2|U|^2}\sum_{t=1}^{|U|}(|\mu_{[x_t]_{\widetilde{R}_A}}| + |1-\nu_{[x_t]_{\widetilde{R}_A}}|)\\
&= \frac{1}{2|U|^2}\sum_{t=1}^{k-1}(|\mu_{[x_t]_{\widetilde{R}_A}}| + |1-\nu_{[x_t]_{\widetilde{R}_A}}|)\\
&\quad + \frac{1}{2|U|^2}(|\mu_{[x_k]_{\widetilde{R}_A}}| + |1-\nu_{[x_k]_{\widetilde{R}_A}}|)\\
&\quad + \frac{1}{2|U|^2}\sum_{t=k+1}^{|U|}(|\mu_{[x_t]_{\widetilde{R}_A}}| + |1-\nu_{[x_t]_{\widetilde{R}_A}}|)
\end{aligned}$$

$$\le \frac{1}{2|U|^2}\sum_{t=1}^{k-1}(|\mu_{[x_t]_{\widetilde{R}_A}}|+|1-\nu_{[x_t]_{\widetilde{R}_A}}|)$$
$$+\frac{1}{2|U|^2}(|\mu_{[x_k]_{\widetilde{R''}_A}}|+|1-\nu_{[x_k]_{\widetilde{R''}_A}}|)$$
$$+\frac{1}{2|U|^2}\sum_{t=k+1}^{|U|}(|\mu_{[x_t]_{\widetilde{R}_A}}|+|1-\nu_{[x_t]_{\widetilde{R}_A}}|)$$
$$=\widetilde{GK}(\widetilde{R'}_A)$$

Thus, $\widetilde{GK}(\widetilde{R}_A)\le\widetilde{GK}(\widetilde{R''}_A)$. □

Corollary 3.5. Let $\widetilde{\mathcal{I}}=(U,AT,V,f,F)$ be an IF $\mathcal{T}$ equivalence information system, $A\subseteq AT$, and $U/\widetilde{R}_A=\{[x_i]_{\widetilde{R}_A}|x_i\in U\}$ be classification of IF $\mathcal{T}$ equivalence relation $\widetilde{R}_A$. If a new knowledge $\widetilde{R''}_A$ can be composed of $\widetilde{R}_A$, then $\widetilde{GK}(\widetilde{R}_A)\le\widetilde{GK}(\widetilde{R''}_A)$.

From the above conclusions, it can be shown that a knowledge granulation provides an important approach to measuring the discernibility ability of a knowledge in IF $\mathcal{T}$ equivalence information systems. The smaller the knowledge granulation is, the stronger its discernibility ability is.

4. Knowledge entropy in IF $\mathcal{T}$ equivalence information systems

In this section, the definitions of knowledge rough entropy and knowledge information entropy will be proposed in IF $\mathcal{T}$ equivalence information systems, and some important properties are investigated.

4.1. Knowledge rough entropy in IF $\mathcal{T}$ equivalence information systems

Definition 4.1. Let $\widetilde{\mathcal{I}}=(U,AT,V,f,F)$ be an IF $\mathcal{T}$ equivalence information system. $A\subseteq AT$, $U/\widetilde{R}_A=\{[x_i]_{\widetilde{R}_A}|x_i\in U\}$ be classification of IF $\mathcal{T}$ equivalence relation $\widetilde{R}_A$. Rough entropy of knowledge $\widetilde{R}_A$, which is denoted by $\widetilde{E_r}(\widetilde{R}_A)$, is defined by

$$\widetilde{E_r}(\widetilde{R}_A)=-\sum_{i=1}^{|U|}\frac{1}{|U|}\log_2\frac{2}{|\mu_{[x_i]_{\widetilde{R}_A}}|+|1-\nu_{[x_i]_{\widetilde{R}_A}}|}.$$

Theorem 4.1. *Let $\widetilde{\mathcal{I}}=(U,AT,V,f,F)$ be an IF $\mathcal{T}$ equivalence information system, $A,B\subseteq AT$, and $U/\widetilde{R}_A=\{[x_i]_{\widetilde{R}_A}|x_i\in U\}$, $U/\widetilde{R}_B=\{[x_i]_{\widetilde{R}_B}|x_i\in U\}$ be classification of two IF $\mathcal{T}$ equivalence relations $\widetilde{R}_A$ and $\widetilde{R}_B$ respectively. We can have the following conclusions.*

(1) *If $|U/\widetilde{R}_A|=|U/\widetilde{R}_B|$, and it exists a bijective map $h:U/\widetilde{R}_A\to U/\widetilde{R}_B$, such that $|[x_i]_{\widetilde{R}_A}|=|h([x_i]_{\widetilde{R}_A})|$, then $\widetilde{E_r}(\widetilde{R}_A)=\widetilde{E_r}(\widetilde{R}_B)$.*
(2) *If $\widetilde{R}_A\preceq\widetilde{R}_B$, then $\widetilde{E_r}(\widetilde{R}_A)\le\widetilde{E_r}(\widetilde{R}_B)$.*
(3) *Rough entropy of knowledge $\widetilde{R}_A$ exists the boundary, i.e., $0\le\widetilde{E_r}(\widetilde{R}_A)\le\log_2|U|$. Where $\widetilde{E_r}(\widetilde{R}_A)=0$ if and only if $\widetilde{R}_A=\widetilde{I}_R$, and $\widetilde{E_r}(\widetilde{R}_A)=\log_2|U|$ if and only if $\widetilde{R}_A=\widetilde{\delta}_R$.*
(4) *If $\widetilde{R}_A$ can be resolved into a new knowledge $\widetilde{R'}_A$, then $\widetilde{E_r}(\widetilde{R'}_A)\le\widetilde{E_r}(\widetilde{R}_A)$.*
(5) *If a new knowledge $\widetilde{R''}_A$ can be composed of $\widetilde{R}_A$, then $\widetilde{E_r}(\widetilde{R}_A)\le\widetilde{E_r}(\widetilde{R''}_A)$.*

Proof. The proofs of them are similar to Theorems 3.1–3.7. □

Example 4.1. (Continued from Example 2.1) By computing, we have that

$$\widetilde{E_r}(\widetilde{R}_{AT})=\frac{1}{5}\log_2 3.8+\frac{1}{5}\log_2 3.75+\frac{1}{5}\log_2 3.925+\frac{1}{5}\log_2 3.975+\frac{1}{5}\log_2 3.3=1.904;$$
$$\widetilde{E_r}(\widetilde{R}_A)=\frac{1}{5}\log_2 3.8+\frac{1}{5}\log_2 3.8+\frac{1}{5}\log_2 4.1+\frac{1}{5}\log_2 4.075+\frac{1}{5}\log_2 3.325=1.930.$$

So, $\widetilde{E_r}(\widetilde{R}_{AT})\le\widetilde{E_r}(\widetilde{R}_A)$.

4.2. Knowledge information entropy in IF $\mathcal{T}$ equivalence information systems

Definition 4.2. Let $\widetilde{\mathcal{I}}=(U,AT,V,f,F)$ be an IF $\mathcal{T}$ equivalence information system. $A\subseteq AT$, $U/\widetilde{R}_A=\{[x_i]_{\widetilde{R}_A}|x_i\in U\}$ be classification of IF $\mathcal{T}$ equivalence relation $\widetilde{R}_A$. Information entropy of knowledge $\widetilde{R}_A$, which is denoted by $\widetilde{E}(\widetilde{R}_A)$, is defined by

$$\widetilde{E}(\widetilde{R}_A)=\sum_{i=1}^{|U|}\frac{1}{|U|}\left(1-\frac{|\mu_{[x_i]_{\widetilde{R}_A}}|+|1-\nu_{[x_i]_{\widetilde{R}_A}}|}{2|U|}\right).$$

Theorem 4.2. *Let $\widetilde{\mathcal{I}} = (U, AT, V, f, \digamma)$ be an IF $\mathcal{T}$ equivalence information system, $A, B \subseteq AT$, and $U/\widetilde{R}_A = \{[x_i]_{\widetilde{R}_A} | x_i \in U\}$, $U/\widetilde{R}_B = \{[x_i]_{\widetilde{R}_B} | x_i \in U\}$ be classification of two IF $\mathcal{T}$ equivalence relation $\widetilde{R}_A$ and $\widetilde{R}_B$ respectively. We can have the following conclusions.*

(1) *If $|U/\widetilde{R}_A| = |U/\widetilde{R}_B|$, and it exists a bijective map $h : U/\widetilde{R}_A \to U/\widetilde{R}_B$, such that $|[x_i]_{\widetilde{R}_A}| = |h([x_i]_{\widetilde{R}_A})|$, then $\widetilde{E}(\widetilde{R}_A) = \widetilde{E}(\widetilde{R}_B)$.*
(2) *If $\widetilde{R}_A \preceq \widetilde{R}_B$, then $\widetilde{E}(\widetilde{R}_A) \geq \widetilde{E}(\widetilde{R}_B)$.*
(3) *Information entropy of knowledge $\widetilde{R}_A$ exists the boundary, i.e., $0 \leq \widetilde{E}(\widetilde{R}_A) \leq 1 - \frac{1}{|U|}$. Where $\widetilde{E}(\widetilde{R}_A) = 1 - \frac{1}{|U|}$ if and only if $\widetilde{R}_A = \widetilde{I}_R$, and $\widetilde{E}(\widetilde{R}_A) = 0$ if and only if $\widetilde{R}_A = \widetilde{\delta}_R$.*
(4) *If $\widetilde{R}_A$ can be resolved into a new knowledge $\widetilde{R'}_A$, then $\widetilde{E}(\widetilde{R'}_A) \geq \widetilde{E}(\widetilde{R}_A)$.*
(5) *If a new knowledge $\widetilde{R''}_A$ can be composed of $\widetilde{R}_A$, then $\widetilde{E}(\widetilde{R}_A) \geq \widetilde{E}(\widetilde{R''}_A)$.*

Proof. The proofs of them are similar to Theorems 3.1–3.7. □

Example 4.2. (Continued from Example 2.1) By computing, we have that

$$\widetilde{E}(\widetilde{R}_{AT}) = \frac{1}{5}[(1-0.76)+(1-0.75)+(1-0.785)+(1-0.795)+(1-0.66)] = 0.25;$$

$$\widetilde{E}(\widetilde{R}_{A}) = \frac{1}{5}[(1-0.76)+(1-0.76)+(1-0.82)+(1-0.815)+(1-0.665)] = 0.236.$$

Thus, we have $\widetilde{E}(\widetilde{R}_{AT}) \geq \widetilde{E}(\widetilde{R}_A)$.

5. Knowledge uncertainly measure in IF $\mathcal{T}$ equivalence information systems

In this section, another uncertainty measure will be introduced, which can provide another important approach to measuring the discernibility ability of a knowledge in IF $\mathcal{T}$ equivalence information systems.

Definition 5.1. Let $\widetilde{\mathcal{I}} = (U, AT, V, f, \digamma)$ be an IF $\mathcal{T}$ equivalence information system. $A \subseteq AT$, $U/\widetilde{R}_A = \{[x_i]_{\widetilde{R}_A} | x_i \in U\}$ be classification of IF $\mathcal{T}$ equivalence relation $\widetilde{R}_A$. Uncertainty measure of knowledge $\widetilde{R}_A$, which is denoted as $\widetilde{E}(\widetilde{R}_A)$, is defined by

$$\widetilde{G}(\widetilde{R}_A) = -\sum_{i=1}^{|U|} \frac{1}{|U|} \log_2 \frac{|\mu_{[x_i]_{\widetilde{R}_A}}| + |1 - \nu_{[x_i]_{\widetilde{R}_A}}|}{2|U|}.$$

Theorem 5.1. *Let $\widetilde{\mathcal{I}} = (U, AT, V, f, \digamma)$ be an IF $\mathcal{T}$ equivalence information system, $A, B \subseteq AT$, and $U/\widetilde{R}_A = \{[x_i]_{\widetilde{R}_A} | x_i \in U\}$, $U/\widetilde{R}_B = \{[x_i]_{\widetilde{R}_B} | x_i \in U\}$ be classification of two IF $\mathcal{T}$ equivalence relation $\widetilde{R}_A$ and $\widetilde{R}_B$ respectively. We can have the following conclusions.*

(1) *If $|U/\widetilde{R}_A| = |U/\widetilde{R}_B|$, and it exists a bijective map $h : U/\widetilde{R}_A \to U/\widetilde{R}_B$, such that $|[x_i]_{\widetilde{R}_A}| = |h([x_i]_{\widetilde{R}_A})|$, then $\widetilde{G}(\widetilde{R}_A) = \widetilde{G}(\widetilde{R}_B)$.*
(2) *If $\widetilde{R}_A \preceq \widetilde{R}_B$, then $\widetilde{G}(\widetilde{R}_A) \geq \widetilde{G}(\widetilde{R}_B)$.*
(3) *Information entropy of knowledge $\widetilde{R}_A$ exists the boundary, i.e., $0 \leq \widetilde{G}(\widetilde{R}_A) \leq \log_2 |U|$. Where $\widetilde{G}(\widetilde{R}_A) = \log_2 |U|$ if and only if $\widetilde{R}_A = \widetilde{I}_R$, and $\widetilde{G}(\widetilde{R}_A) = 0$ if and only if $\widetilde{R}_A = \widetilde{\delta}_R$.*
(4) *If $\widetilde{R}_A$ can be resolved into a new knowledge $\widetilde{R'}_A$, then $\widetilde{G}(\widetilde{R'}_A) \geq \widetilde{G}(\widetilde{R}_A)$.*
(5) *If a new knowledge $\widetilde{R''}_A$ can be composed of $\widetilde{R}_A$, then $\widetilde{G}(\widetilde{R}_A) \geq \widetilde{G}(\widetilde{R''}_A)$.*

Proof. The proof of them are similar to Theorems 3.1–3.7. □

Example 5.1. (Continued from Example 2.1) By computing, we have that

$$\widetilde{G}(\widetilde{R}_{AT}) = -\frac{1}{5}[(\log_2(0.76)) + (\log_2(0.75)) + (\log_2(0.785)) + (\log_2(0.795)) + (\log_2(0.66))] = 0.418;$$

$$\widetilde{G}(\widetilde{R}_{A}) = -\frac{1}{5}[(\log_2(0.76)) + (\log_2(0.76)) + (\log_2(0.82)) + (\log_2(0.815)) + (\log_2(0.665))] = 0.392.$$

Thus, we have $\widetilde{G}(\widetilde{R}_{AT}) \geq \widetilde{G}(\widetilde{R}_A)$.

6. Relationships among knowledge granulation, knowledge entropy and uncertainty measure

In this section, we will discuss the relationships among knowledge granulation, knowledge entropy and uncertainty measure.

Theorem 6.1. *Let $\widetilde{\mathcal{I}} = (U, AT, V, f, \digamma)$ be an IF $\mathcal{T}$ equivalence information system. $A \subseteq AT$, $U/\widetilde{R}_A = \{[x_i]_{\widetilde{R}_A} | x_i \in U\}$ be classification of IF $\mathcal{T}$ equivalence relation $\widetilde{R}_A$. The relation of knowledge granulation $\widetilde{GK}(\widetilde{R}_A)$ and information entropy $\widetilde{E}(\widetilde{R}_A)$ of knowledge $\widetilde{R}_A$ is*

$$\widetilde{GK}(\widetilde{R}_A) + \widetilde{E}(\widetilde{R}_A) = 1.$$

Proof. Since $U/\widetilde{R}_A = \{[x_i]_{\widetilde{R}_A} | x_i \in U\}$ is classification of IF $\mathcal{T}$ equivalence relation $\widetilde{R}_A$, we have

$$\begin{aligned}
&\widetilde{GK}(\widetilde{R}_A) + \widetilde{E}(\widetilde{R}_A) \\
&= \frac{1}{2|U|^2} \sum_{i=1}^{|U|} (|\mu_{[x_i]_{\widetilde{R}_A}}| + |1 - \nu_{[x_i]_{\widetilde{R}_A}}|) \\
&\quad + \sum_{i=1}^{|U|} \frac{1}{|U|} \left(1 - \frac{|\mu_{[x_i]_{\widetilde{R}_A}}| + |1 - \nu_{[x_i]_{\widetilde{R}_A}}|}{2|U|}\right) \\
&= \sum_{i=1}^{|U|} \left(\frac{|\mu_{[x_i]_{\widetilde{R}_A}}| + |1 - \nu_{[x_i]_{\widetilde{R}_A}}|}{2|U|^2}\right) \\
&\quad + \sum_{i=1}^{|U|} \left(\frac{1}{|U|} - \frac{|\mu_{[x_i]_{\widetilde{R}_A}}| + |1 - \nu_{[x_i]_{\widetilde{R}_A}}|}{2|U|^2}\right) \\
&= \sum_{i=1}^{|U|} \frac{1}{|U|} = 1. \qquad \square
\end{aligned}$$

Example 6.1. (Continued from Example 3.1 and Example 4.2) In Examples 3.1 and 4.2, we have acquired that

$$\widetilde{GK}(\widetilde{R}_{AT}) = 0.75, \widetilde{GK}(\widetilde{R}_A) = 0.764;$$
$$\widetilde{E}(\widetilde{R}_{AT}) = 0.25, \widetilde{E}(\widetilde{R}_A) = 0.236.$$

So, the following is obvious

$$\widetilde{GK}(\widetilde{R}_{AT}) + \widetilde{E}(\widetilde{R}_{AT}) = 1,$$

$$\widetilde{GK}(\widetilde{R}_A) + \widetilde{E}(\widetilde{R}_A) = 1.$$

Theorem 6.2. *Let $\widetilde{\mathcal{I}} = (U, AT, V, f, \digamma)$ be an IF $\mathcal{T}$ equivalence information system. $A \subseteq AT$, $U/\widetilde{R}_A = \{[x_i]_{\widetilde{R}_A} | x_i \in U\}$ be classification of IF $\mathcal{T}$ equivalence relation $\widetilde{R}_A$. Relationship between uncertainty measure $\widetilde{G}(\widetilde{R}_A)$ and rough entropy $\widetilde{E_r}(\widetilde{R}_A)$ of knowledge $\widetilde{R}_A$ is*
$\widetilde{G}(\widetilde{R}_A) + \widetilde{E_r}(\widetilde{R}_A) = \log_2 |U|$.

Proof. Because $U/\widetilde{R}_A = \{[x_i]_{\widetilde{R}_A} | x_i \in U\}$ is classification of IF $\mathcal{T}$ equivalence relation $\widetilde{R}_A$, we have

$$\begin{aligned}
&\widetilde{G}(\widetilde{R}_A) + \widetilde{E_r}(\widetilde{R}_A) \\
&= -\sum_{i=1}^{|U|} \frac{1}{|U|} \log_2 \frac{|\mu_{[x_i]_{\widetilde{R}_A}}| + |1 - \nu_{[x_i]_{\widetilde{R}_A}}|}{2|U|} \\
&\quad + \left(-\sum_{i=1}^{|U|} \frac{1}{|U|} \log_2 \frac{2}{|\mu_{[x_i]_{\widetilde{R}_A}}| + |1 - \nu_{[x_i]_{\widetilde{R}_A}}|}\right) \\
&= -\sum_{i=1}^{|U|} \frac{1}{|U|} [\log_2(|\mu_{[x_i]_{\widetilde{R}_A}}| + |1 - \nu_{[x_i]_{\widetilde{R}_A}}|) \\
&\quad - \log_2(2|U|)] \\
&\quad - \sum_{i=1}^{|U|} \frac{1}{|U|} [1 - \log_2(|\mu_{[x_i]_{\widetilde{R}_A}}| + |1 - \nu_{[x_i]_{\widetilde{R}_A}}|)] \\
&= \sum_{i=1}^{|U|} \frac{1}{|U|} (\log_2(2|U|) - 1) \\
&= \sum_{i=1}^{|U|} \frac{1}{|U|} (\log_2(|U|)) \\
&= \log_2(|U|). \qquad \square
\end{aligned}$$

Example 6.2. (Continued from Example 5.1 and Example 4.1) In Examples 5.1 and 4.1, we have acquired that

$$\widetilde{G}(\widetilde{R}_{AT}) = 0.418, \quad \widetilde{G}(\widetilde{R}_A) = 0.392;$$

$$\widetilde{E_r}(\widetilde{R}_{AT}) = 1.904. \quad \widetilde{E_r}(\widetilde{R}_A) = 1.930.$$

So, the following is obvious

$$\widetilde{G}(\widetilde{R}_{AT}) + \widetilde{E_r}(\widetilde{R}_{AT}) = 2.322 = \log_2 |U|,$$

$$\widetilde{G}(\widetilde{R}_A) + \widetilde{E_r}(\widetilde{R}_A) = 2.322 = \log_2 |U|.$$

7. Uncertainty measure of rough sets in IF $\mathcal{T}$ equivalence information systems

In this section, we will introduce the definitions of roughness measure and accuracy measure of rough sets in IF $\mathcal{T}$ equivalence information systems by generalizing the classical rough degree of Pawlak rough set, and through two illustrative examples,we find the limitations of roughness measure and accuracy measure for evaluating uncertainty of a set and approximation accuracy of a rough classification in IF $\mathcal{T}$ equivalence information systems. In order to overcome the limitations, the concept of rough entropy will be proposed in IF $\mathcal{T}$ equivalence information systems.

Definition 7.1. Let $\widetilde{\mathcal{I}} = (U, AT, V, f, \digamma)$ be an IF $\mathcal{T}$ equivalence information system. $A \subseteq AT$, $\widetilde{R}_A$ be an IF $\mathcal{T}$ equivalence relation, The roughness measure of $\widetilde{X}$ is defined by

$$\rho_{\widetilde{R}_A}(\widetilde{X}) = 1 - \frac{|\mu_{\underline{\widetilde{R}_A}(\widetilde{X})}| + |1 - \nu_{\underline{\widetilde{R}_A}(\widetilde{X})}|}{|\mu_{\overline{\widetilde{R}_A}(\widetilde{X})}| + |1 - \nu_{\overline{\widetilde{R}_A}(\widetilde{X})}|},$$

where $\widetilde{X} \neq \widetilde{1}_\emptyset$ and $|\cdot|$ denotes the cardinality of an IF set. For convenience, we preside $\rho_{\widetilde{R}_A}(\widetilde{1}_\emptyset) = 1$.

The accuracy measure of $\widetilde{X}$ is defined by

$$\varrho_{\widetilde{R}_A}(\widetilde{X}) = 1 - \rho_{\widetilde{R}_A}(\widetilde{X}) = \frac{|\mu_{\underline{\widetilde{R}_A}(\widetilde{X})}| + |1 - \nu_{\underline{\widetilde{R}_A}(\widetilde{X})}|}{|\mu_{\overline{\widetilde{R}_A}(\widetilde{X})}| + |1 - \nu_{\overline{\widetilde{R}_A}(\widetilde{X})}|}.$$

Example 7.1. (Continued form Example 2.1) In Example 2.1, we have known $U/\widetilde{R}_{AT} \subseteq U/\widetilde{R}_A$, i.e., classification $U/\widetilde{R}_{AT}$ is finer than classification $U/\widetilde{R}_A$ in the system. Consider the IF t-norm $\mathcal{T}$: $\mathcal{T}(\widehat{\alpha}, \widehat{\beta}) = (T(\alpha_1, \beta_1),\ S(\alpha_2, \beta_2))$, where $\widehat{\alpha} = (\alpha_1, \alpha_2)$, $\widehat{\beta} = (\beta_1, \beta_2)$, $T(\alpha_1, \beta_1) = \max\{0, \alpha_1 + \beta_1 - 1\}$, $S(\alpha_2, \beta_2) = \min\{1, \alpha_2 + \beta_2\}$. for $\widetilde{X'} = \{(0.2, 0.7), (0.4, 0.5), (0.5, 0.2), (0.7, 0.2), (1.0, 0)\}$. Then we can calculate the $\overline{\widetilde{R}_{AT}}(\widetilde{X'})$, $\underline{\widetilde{R}_{AT}}(\widetilde{X'})$, $\overline{\widetilde{R}_A}(\widetilde{X'})$ and $\underline{\widetilde{R}_A}(\widetilde{X'})$ as follows:

$$\begin{aligned}\overline{\widetilde{R}_{AT}}(\widetilde{X'}) = \overline{\widetilde{R}_A}(\widetilde{X'}) &= \{(0.5, 0.4), (0.4, 0.4), (0.6, 0.2),\\ &\quad (0.7, 0.2), (1.0, 0)\};\\ \underline{\widetilde{R}_{AT}}(\widetilde{X'}) = \underline{\widetilde{R}_A}(\widetilde{X'}) &= \{(0.2, 0.7), (0.35, 0.5), (0.4, 0.4),\\ &\quad (0.45, 0.4), (0.7, 0.2)\}.\end{aligned}$$

Thus, by calculating, the rough degrees of $\widetilde{X'}$ about knowledge $\widetilde{R}_{AT}$ and $\widetilde{R}_A$ can be obtained respectively, which are $\rho_{\widetilde{R}_{AT}}(\widetilde{X'}) = \rho_{\widetilde{R}_{AT}}(\widetilde{X'}) = \frac{20}{71}$.

In other words, the uncertainty of knowledge $\widetilde{R}_A$ is larger than that of $\widetilde{R}_{AT}$ in Example 2.1, but $\widetilde{X'}$ has the same rough degree. Therefore, it is necessary to find a new and more accurate uncertainty measure for rough sets in IF $\mathcal{T}$ equivalence information systems.

In the next, the concept of rough entropy will be proposed, and it will be shown that it is a new and more accurate uncertainty measure for rough sets in IF $\mathcal{T}$ equivalence information systems.

Definition 7.2. Let $\widetilde{\mathcal{I}} = (U, AT, V, f, \digamma)$ be an IF $\mathcal{T}$ equivalence information system. $A \subseteq AT$, $\widetilde{R}_A$ be an IF $\mathcal{T}$ equivalence relation, the rough entropy of $\widetilde{X}$ about knowledge $\widetilde{R}_A$ is defined as follows:
$E_{\widetilde{R}_A}(\widetilde{X}) = \rho_{\widetilde{R}_A}(\widetilde{X}) \cdot \widetilde{GK}(\widetilde{R}_A)$.

Furthermore, the following property can be obtained about the entropy of rough sets.

Theorem 7.1. *Let $\widetilde{\mathcal{I}} = (U, AT, V, f, \digamma)$ be an IF $\mathcal{T}$ equivalence information system, $A, B \subseteq AT$, and $U/\widetilde{R}_A = \{[x_i]_{\widetilde{R}_A} | x_i \in U\}$, $U/\widetilde{R}_B = \{[x_i]_{\widetilde{R}_B} | x_i \in U\}$ be classification of two IF $\mathcal{T}$ equivalence relation $\widetilde{R}_A$ and $\widetilde{R}_B$ respectively. We can have the following conclusions.*

(1) *If $|U/\widetilde{R}_A| = |U/\widetilde{R}_B|$, and it exists a bijective map $h : U/\widetilde{R}_A \to U/\widetilde{R}_B$, such that $|[x_i]_{\widetilde{R}_A}| = |h([x_i]_{\widetilde{R}_A})|$, then $E_{\widetilde{R}_A}(\widetilde{X}) = E_{\widetilde{R}_B}(\widetilde{X})$.*
(2) *If $\widetilde{R}_A \preceq \widetilde{R}_B$, then $E_{\widetilde{R}_A}(\widetilde{X}) \le E_{\widetilde{R}_B}(\widetilde{X})$.*
(3) *Rough entropy of $\widetilde{X}$ about knowledge $\widetilde{R}_A$ exists the boundary, i.e., $0 \le E_{\widetilde{R}_A}(\widetilde{X}) \le 1$. Where $E_{\widetilde{R}_A}(\widetilde{X}) = 0$ if and only if $\widetilde{R}_A = \widetilde{I}_R$, and $E_{\widetilde{R}_A}(\widetilde{X}) = 1$ if and only if $\widetilde{R}_A = \widetilde{\delta}_R$*
(4) *If $\widetilde{R}_A$ can be resolved into a new knowledge $\widetilde{R'}_A$, then $E_{\widetilde{R'}_A}(\widetilde{X}) \le E_{\widetilde{R}_A}(\widetilde{X})$.*
(5) *If a new knowledge $\widetilde{R''}_A$ can be composed of $\widetilde{R}_A$, then $E_{\widetilde{R}_A}(\widetilde{X}) \le E_{\widetilde{R''}_A}(\widetilde{X})$.*

Proof. The proofs of them can be acquired directly by Theorems 3.1–3.7 and Definition 7.1. □

From the above, the rough entropy of rough sets is related not only to its own rough degree, but also to the uncertainty of knowledge in IF $\mathcal{T}$ equivalence information systems.

Example 7.2. (Continued form Example 7.1) The rough entropy of $\widetilde{X'}$ in Example 7.1 is calculated about knowledge $\widetilde{R}_{AT}$ and $\widetilde{R}_A$, respectively, which are

$$\begin{aligned}E_{\widetilde{R}_{AT}}(\widetilde{X'}) &= \rho_{\widetilde{R}_{AT}}(\widetilde{X'}) \cdot \widetilde{GK}(\widetilde{R}_{AT})\\ &= 0.75 \times 0.82 = 0.615,\end{aligned}$$

$$E_{\widetilde{R}_A}(\widetilde{X}') = \rho_{\widetilde{R}_A}(\widetilde{X}') \cdot \widetilde{GK}(\widetilde{R}_A) = 0.764 \times 0.82 = 0.626.$$

Thus, we have

$$E_{\widetilde{R}_{AT}}(\widetilde{X}') < E_{\widetilde{R}_A}(\widetilde{X}').$$

8. Conclusion

Atanassov's intuitionistic fuzzy $\mathcal{T}$ equivalence information systems are more material and concise to describe the essence of fuzziness. Thus, the uncertainty measure method of knowledge is one of the most important research tasks in Atanassov's intuitionistic fuzzy $\mathcal{T}$ equivalence information systems. In this paper, we considered the binary relation and the boundary of a rough set of the uncertainty of knowledge from two aspects. In binary relation aspect, we introduced the concepts of knowledge granulation, knowledge entropy and knowledge uncertainty measure in Atanassov's intuitionistic fuzzy $\mathcal{T}$ equivalence information systems, and discussed some important properties of them. From these properties, it can be shown that these measures provided some important approaches to measuring the discernibility ability of different knowledges in Atanassov's intuitionistic fuzzy $\mathcal{T}$ equivalence information systems. In boundary of a rough set aspect, we introduced a natural extension of classical rough degree to measure the roughness of rough sets in Atanassov's intuitionistic fuzzy $\mathcal{T}$ equivalence information systems, and by an example, we found the limitations of roughness measure and accuracy measure for evaluating uncertainty of a set and approximation accuracy of a rough classification. Thus the concept of rough entropy was proposed in Atanassov's intuitionistic fuzzy $\mathcal{T}$ equivalence information systems, which was more accurate than natural extension of classical rough degree to measure the roughness of rough sets. These results will be helpful for understanding of the essence of uncertainty measure in Atanassov's intuitionistic fuzzy $\mathcal{T}$ equivalence information systems.

Acknowledgements

This work is supported by National Natural Science Foundation of China (No. 61105041, 71071124 and 11001227), Postdoctoral Science Foundation of China (No. 20100481331), Science and Technology Program of Board of Education of Chongqing (KJ120805), National Natural Science Foundation of CQ CSTC (No. cstc2011jjA40037), and Graduate Innovation Foundation of Chongqing University of Technology (No. YCX2011312).

References

[1] K. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **20** (1986), 87–96.
[2] T. Beaubouef, F.E. Petry and G. Arora, Information-theoretic measures of uncertainty for rough sets and rough relational databases, *Information Sciences* **109** (1998), 185C–195.
[3] H. Bustince and P. Burillo, Structures on intuitionistic fuzzy relations, *Fuzzy Sets and Systems* **7** (1996), 293–303.
[4] D. Coker, Rough sets are intuitionistic L-fuzzy sets, *Fuzzy Sets and Systems* **96** (1998), 381–383.
[5] K. Chakrabarty, T. Gedeon and L. Koczy, Intuitionistic fuzzy rough set, in: *Proceedings of Fourth Joint Conference on Information Sciences (JCIS)*, Durham, NC, 1998, 211–214.
[6] C. Cornelis, M.D. Cock and E.E. Kerre, Intuitionistic fuzzy rough sets: At the crossroads of imperfect knowledge, *Expert Systems* **20** (2003), 260–270.
[7] C. Cornelis, G. Deschrijver and E.E. Kerre, Implication in intuitionistic fuzzy and interval-valued fuzzy set theory: Construction classification, application. *International Journal of Approximate Reasoning* **35** (2004), 55–95.
[8] E.E. Castinira, C. Torres-Blanc and S. Cubillo, Measuring contradiction on A-IFS defined in ?nite universes, *Knowledge-Based Systems* **24** (2011), 1297C–1309.
[9] G. Deschrijver and E.E. Kerre, On the position of intuitionistic fuzzy set theory in the framework of theories modelling imprecision, *Information Sciences* **177** (2007), 1860–1866.
[10] I. Duentsch and G. Gediga, Uncertainty measures of rough set prediction, *Artificial Intelligence* **106** (1998), 109–137.
[11] L. Dymova and P. Sevastjanov, An interpretation of intuitionistic fuzzy sets in terms of evidence theory: Decision making aspect, *Knowledge-Based Systems* **23** (2010), 772–782.
[12] S.P. Jena and S.K. Ghosh, Intuitionistic fuzzy rough sets, *Notes on Intuitionistic Fuzzy Sets* **8** (2002), 1–18.
[13] L. Lin, X.H. Yuan and Z.Q. Xia, Multicriteria fuzzy decision-making methods based on intuitionistic fuzzy sets, *Journal of Computer and System Sciences* **73** (2007), 84–88.
[14] J.Y. Liang, K.S. Chin, C. Dang and R.C.M. Yam, A new method for measuring uncertainty and fuzziness in rough set theory, *International Journal of General Systems* **31** (2002), 331–342.
[15] J.Y. Liang and Z.Z. Shi, The information entropy, rough entropy and knowledge granulation in rough set theory, *International Journal of Uncertainty Fuzziness and Knowledge-Based Systems* **12** (2004), 37–46.
[16] J.Y. Liang, Z.Z. Shi, D.Y. Li and M.J. Wierman, Information entropy, rough entropy and knowledge granulation in incomplete information systems, *International Journal of General Systems* **35** (2006), 641–654.
[17] S. Nanda and S. Majumda, Fuzzy rough sets, *Fuzzy Sets and Systems* **45** (1992), 157–160.
[18] Z. Pei and L. Zheng, A novel approach to multi-attribute decision making based on intuitionistic fuzzy sets, *Expert Systems with Applications* **39** (2012), 2560–2566.
[19] Z. Pawlak, Rough sets, *International Journal of Computer and Information Science* **11** (1982), 341–356.
[20] Z. Pawlak, *Rough, Sets: Theoretical Aspects of Reasoning about Data*, Kluwer Academic Publisher, Boston, 1991.

[21] Y.H. Qian and J.Y. Liang, Combination entropy and combination granulation in rough set theory, *International Journal of Uncertainty, Fuzziness and Knowledge-Based System* **16** (2008), 179–193.

[22] Y.H. Qian, J.Y. Liang, W.Z. Wu and C.Y. Dang, Knowledge structure, knowledge granulation and knowledge distance in a knowledge base, *International Journal of Approximate Reasoning* **50** (2009), 174–188.

[23] Y.H. Qian, J.Y. Liang and C.Y. Dang, Fuzzy information granularity in a binary granular structure, *IEEE Transactions on Fuzzy Systems* **19** (2011), 253–264.

[24] A.M. Radzikowska, Rough approximation operations based on IF sets, in: *ICAISC 2006, Lecture Notes in Computer Science*, vol. 4029, Springer, Berlin, 2006, 528–537.

[25] S. Rizvi, H.J. Naqvi and D. Nadeem, Rough intuitionistic fuzzy set, in: *Proceedings of the Sixth Joint Conference on Information Sciences (JCIS)*, Durham, NC, 2002, 101–104.

[26] S.K. Samanta and T.K. Mondal, Intuitionistic fuzzy rough sets and rough intuitionistic fuzzy sets, *Journal of Fuzzy Mathematics* **9** (2001), 561–582.

[27] C. Shannon, The mathematical theory of communication, *Bell System Technical Journal* **27** (1948), 373–423.

[28] L.K. Vlachos and G.D. Sergiadis, Intuitionistic fuzzy information - applications to pattern recognition, *Pattern Recognition Letters* **28** (2007), 197–206.

[29] M. Wierman, Measuring uncertainty in rough set theory, *International Journal of General Systems* **28** (1999), 283–297.

[30] C.P. Wei, P. Wang and Y.Z. Zhang, Entropy, similarity measure of interval-valued intuitionistic fuzzy sets and their applications, *Information Sciences* **181** (2011), 4273–4286.

[31] Z.S. Xu, Intuitionistic preference relations and their application in group decision making, *Information Sciences* **177** (2007), 2363–2379.

[32] W.H. Xu, X.Y. Zhang and W.X. Zhang, Knowledge granulation, knowledge entropy and knowledge uncertainty measure in ordered information systems, *Applied Soft Computing* **9** (2009), 1244–1251.

[33] Z.S. Xu, Intuitionistic fuzzy aggregation operators, *IEEE Transactions on Fuzzy Systems* **15** (2007), 1179–1187.

[34] Z.S. Xu and R.R. Yanger, Some geometric aggregation operators based intuitionistic fuzzy sets, *International Journal of General Systems* **35** (2006), 417–433.

[35] E. Szmidt and J. Kacprzyk, Distances between intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **114** (2000), 505–518.

[36] Z.B. Xu, Some similarity measures of intuitionistic fuzzy sets and their applications to multiple attribute decision making, *Fuzzy Optimization and Decision Making* **6** (2007), 106–121.

[37] L.A. Zadeh, Fuzzy sets, *Information and Control* **8** (1965), 338–353.

[38] L. Zadeh, Fuzzy sets and information granularity, *Advances in Fuzzy Set Theory and Applications* **1** (1979), 3–18.

[39] W.X. Zhang and G.J. Wang, *Fuzzy Mathematics*, Xi'an Jiaotong University Press, Xi'an, 1991.

[40] S.F. Zhang and S.Y. Liu, A GRA-based intuitionistic fuzzy multi-criteria group decision making method for personnel selection, *Expert Systems with Applications* **38** (2011), 11401–11405.